Exercise 4

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' - 2y' + 2y = x + e^x$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' + 2y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - 2(re^{rx}) + 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 2r + 2 = 0$$

Solve for r.

$$r = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

$$r = \{1 - i, 1 + i\}$$

Two solutions to the ODE are $e^{(1-i)x}$ and $e^{(1+i)x}$. By the principle of superposition, then,

$$\begin{aligned}
y_c(x) &= C_1 e^{(1-i)x} + C_2 e^{(1+i)x} \\
&= C_1 e^x e^{-ix} + C_2 e^x e^{ix} \\
&= e^x (C_1 e^{-ix} + C_2 e^{ix}) \\
&= e^x [C_1(\cos x - i \sin x) + C_2(\cos x + i \sin x)] \\
&= e^x [(C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x] \\
&= e^x (C_3 \cos x + C_4 \sin x).
\end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - 2y_p' + 2y_p = x + e^x \tag{2}$$

Since the inhomogeneous term is the sum of a polynomial of degree 1 and an exponential function, the particular solution is $y_p = Ax + B + Ce^x$.

$$y_p = Ax + B + Ce^x \quad \rightarrow \quad y_p' = A + Ce^x \quad \rightarrow \quad y_p'' = Ce^x$$

Substitute these formulas into equation (2).

$$Ce^x - 2(A + Ce^x) + 2(Ax + B + Ce^x) = x + e^x$$

$$(-2A + 2B) + (2A)x + (C - 2C + 2C)e^x = x + e^x$$

Match the coefficients on both sides to get a system of equations for A, B, and C.

$$\left.\begin{aligned} -2A + 2B &= 0 \\ 2A &= 1 \\ C - 2C + 2C &= 1 \end{aligned}\right\}$$

Solving it yields

$$A = \frac{1}{2} \quad \text{and} \quad B = \frac{1}{2} \quad \text{and} \quad C = 1,$$

which means the particular solution is

$$y_p = \frac{1}{2}x + \frac{1}{2} + e^x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= e^x(C_3 \cos x + C_4 \sin x) + \frac{1}{2}x + \frac{1}{2} + e^x, \end{aligned}$$

where C_3 and C_4 are arbitrary constants.